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Constructing Flexible Dual Contingency Plans for Optimal Linear Designs with Multiple Criteria

YONG SHI*

College of Business Administration, University of Nebraska at Omaha, Omaha, Nebraska 68182

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Given a linear design problem with multiple objectives and multiple resource availability levels, we first use the multi-criteria and multi-constraint level (MC^2) simplex method to identify a set of potentially good designs and to construct their corresponding contingency plans for coping with changes of resource availability levels and the unit contribution of selected products (opportunities). Then we use known techniques of decisionmaking under uncertainty to select the optimal design(s) from these potentially good designs and their contingency plans as the final decision. For some change of the unit contribution of selected products in a potentially good design, the potentially good design may not satisfy the optimality condition under consideration. To overcome the non-optimal situation, we must construct dual contingency plans to convert non-optimal solutions into optimal ones for the potentially good design. This paper discusses how to construct flexible dual contingency plans for potentially good designs in which the number of selected products made, the contribution of the products, and units of slack resources are flexibly adjusted to meet the optimality. Based on theoretical results, we propose a method of effectively and systematically identifying all potentially good designs and the corresponding flexible dual contingency plans under various decision situations.

1. INTRODUCTION

Traditional optimal design problems are usually formulated by mathematical programming which involves a single objective function and a single resource availability level (see [7, 14]). This approach intends to find an optimal solution for a given system. It, however, does not consider

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two important aspects of formulating and solving optimal design problems: designing optimal systems and constructing related contingency plans for each optimal system to cope with various decision situations.

Zeleny [16, 17] first emphasized the importance of designing optimal systems and used multi-objective linear programming to formulate optimal linear design problems and to design the optimal systems. But Zeleny's approach does not explicitly address contingency plans for different decision situations. Later, Lee *et al.* [6] proposed a basic procedure to design the optimal systems and construct their corresponding contingency plans by using multi-objective and multi-constraint level (MC^2) linear programming derived by Seiford and Yu [8].

According to the basic procedure of [6], one first uses MC^2 linear programming to formulate the design problems. Then one uses the MC^2-simplex method [8] to identify a *set* of potentially good designs (PGDs) that contain selected products (opportunities) and can potentially optimize the given design problem under certain ranges of decision parameters, such as the contribution of objectives and resource availability levels. Third, for each PGD one uses submodels of the design model to construct rigid primal contingency plans (RPCP) to overcome changes of the decision parameters. A RPCP for a PGD which contains some products and slack resources selected in the PGD and purchased external resources can convert infeasible solutions into feasible ones. Finally, using known techniques of decision making under uncertainty, one selects the optimal design(s) from the set of PGDs and their contingency plans as the final decision.

Being motivated by [6], Shi and Yu [11] explored the relationship between the "shadow price" of a PGD and the corresponding RPCPs for the PGD. Shi and Yu [12] developed different submodels of the design model for constructing flexible primal contingency plans (FPCP) which can contain not only some products and slack resources selected in a PGD and the external resources, but also some slack resources that are not in the PGD. The FPCPs in turn provide flexibility for the decision maker to utilize slack (unused) resources. Shi *et al.* [13] suggested taking unions of given PGDs for generating good designs (GGD) as candidates for the optimal linear designs. They showed that the GGDs can relax the assumption imposed by the MC^2-simplex method to compute the PGDs (see [6]) and are preferred to or equivalent to the PGDs in terms of the optimality condition. Most recently, Shi [10] studied how to use the dual model of the linear design model to construct rigid dual contingency plans (RDCP) for a PGD. For some change of the unit contribution of selected products in a PGD, the PGD may not satisfy the optimality condition under consideration. To overcome the non-optimal situation, this duality approach *adjusts* the units of selected products made and the unit contribution of the products in PGD to ensure the optimality condition for the PGD. Thus the RDCPs can convert non-optimal solutions into optimal solutions for the PGD. Nevertheless, the flexibility of how to utilize all possible slack resources in constructing the dual contingency plans has not been discussed.

In this paper, we explore how to construct flexible dual contingency plans (FDCP) for a given PGD, in which the number of selected products made, the contribution of the products, and the units of all possible slack resources are flexibly adjusted to meet the optimality. We shall begin with a brief review of a mathematical model for an optimal linear design problem with multiple objectives and multiple resource availability levels and how to use the MC^2-simplex method to identify a set of PGDs. Then we focus on how to develop submodels for constructing the corresponding FDCPs for each PGD by adjusting the units of the selected products to make them become optimal with respect to certain ranges of the decision parameters. We also describe an augmented dual model for locating all FDCPs for each PGD for the given optimal linear design problem. Based on theoretical results, we propose a method of effectively and systematically identifying all PGDs and the corresponding FDCPs under various decision situations. A numerical example is used to illustrate this method. Finally, we conclude this paper with some remaining research problems.

2. A Mathematical Model and Potentially Good Designs

We sketch a known mathematical model of optimal linear design problems with multiple objectives and multiple resource availability levels as follows. (For the details, the reader can refer to [6, 9, 11, 12]).

Let $N = \{1, \ldots, n\}$ be n products (opportunities) under consideration. The model of selecting optimal linear designs with multiple objectives and multiple resource availability levels can be formulated by

$$\begin{aligned} & \text{Max } \lambda^t Cx \\ \text{s.t.} \quad & Ax \leq D\gamma \qquad (1) \\ & x \geq 0, \end{aligned}$$

where $C \in \mathbf{R}^{q \times n}$ is the contribution matrix whose q rows are the coefficients of q objectives; $A \in \mathbf{R}^{m \times n}$ is the unit consumption matrix of resources; and $D \in \mathbf{R}^{m \times p}$ is the matrix of resource availability levels whose p columns are p resource availability levels; $x \in \mathbf{R}^n$ is the product variables; and both γ, called the *resource parameter*, and λ, called the *contribution parameter*, are normalized; that is,

$$\gamma \in \mathbf{R}^{p+} \text{ with } \gamma_k > 0 \text{ and } \sum_{k=1}^{p} \gamma_k = 1 \quad \text{and}$$

$$\lambda \in \mathbf{R}^{q+} \text{ with } \lambda_k > 0 \text{ and } \sum_{k=1}^{q} \lambda_k = 1.$$

From model (1), we see that if parameters (γ, λ) are known ahead of the design time, then the design problem reduces to the traditional model with single objective and single resource availability level. One can select the best k products from possible n products as the optimal design by using linear programming (see, e.g., Charnes and Cooper [2] and Dantzig [4]). If parameter γ is known and parameter λ is unknown, then we can use multiobjective linear programming to design the optimal systems (see [16, 17]). In this paper we assume that parameters (γ, λ) are not known ahead of time. Under this assumption, the change of γ may make the original choice infeasible, while the change of λ can render the choice not optimal. Thus, contingency plans must be constructed to overcome the difficult decision situations.

By adding slack variables s (note that we let the contribution of slack variables be zero) into the constraints of model (1), it becomes

$$\begin{aligned} & \text{Max } \lambda' C x \\ \text{s.t.} \quad & Ax + I_m s = D\gamma \\ & x \geq 0, \end{aligned}$$

where $s \in \mathbf{R}^m$ is an m-dimensional vector and I_m is an $(m \times m)$ identity matrix.

Then the initial simplex table can be written as

x	s	RHS	
A	I_m	$D\gamma$	(2)
$-\lambda' C$	0	0	(3)

In (2) and (3) the first and second blocks of columns are the coefficients associated respectively with the original and slack variables. Equation (2) represents constraints while Eq. (3) is the objective functions.

Let J be the index set of basic variables (without confusion, J is also called a *basis*). Given a J with the basic variables denoted by $x(J)$, we define the *associated basis matrix* B_J as the submatrix of $[A, I_m]$ in (2) with column index in J (i.e., column j of $[A, I_m]$ is in B_J iff $j \in J$) and the *associated objective function coefficient* C_J as the submatrix of $[C, 0]$ in (3) with column index in J. Let $x(J')$ be the non-basic variables corresponding to given $x(J)$. Then we rearrange the index, if necessary, and decompose $[A, I_m]$ into $[B_J, R]$, where R is the submatrix of $[A, I_m]$ associated with $x(J')$. The table (2), (3) can be rewritten as

$x(J)$	$x(J')$	RHS	
B_J	R	$D\gamma$	(4)
$-\lambda' C_J$	$-\lambda' C_{J'}$	0	(5)

where $C_{J'}$ is the submatrix of $[C, 0]$ associated with $x(J')$.

Table (4), (5) can be further rewritten as

$x(J)$	$x(J')$	RHS	
I_m	$B_J^{-1}R$	$B_J^{-1}D\gamma$	(6)
0	$\lambda'[C_J B_J^{-1}R - C_{J'}]$	$\lambda' C_J B_J^{-1} D\gamma$	(7)

Table (6), (7) is an MC^2-simplex tableau with B_J as the basis, and

$$(6) = B_J^{-1} \cdot (4)$$

(i.e., premultiply (4) by B_J^{-1}), and

$$(7) = \lambda' C_J \cdot (6) + (5).$$

Table (6), (7) implies that $x(J, \gamma) = B_J^{-1} D\gamma$ is a basic solution associated with (J, γ) and its objective value is given by $\lambda' C_J B_J^{-1} D\gamma$ when λ is specified.

DEFINITION 1. Given a basis J for model (1), define its corresponding

(i) *primal parameter set* by

$$\Gamma_1(J) = \{\gamma \geq 0 | B_J^{-1} D\gamma \geq 0\};$$

and its

(ii) *dual parameter set* by

$$\Lambda_1(J) = \{\lambda \geq 0 | \lambda'[C_J B_J^{-1} R - C_{J'}] \geq 0\}.$$

The following is well known (see Chap. 8 of [15] for details).

STATEMENT 1. (i) *The resulting solution* $x(J, \gamma) = B_J^{-1}D$, $\gamma \geq 0$, *and* J *is a feasible basis iff* $\gamma \in \Gamma_1(J)$.

(ii) *The solution* $x(J, \gamma)$ *is optimal iff* $\gamma \in \Gamma_1(J)$ *and* $\lambda \in \Lambda_1(J)$.

(iii) J *is a primal potential basis iff* $\Gamma_1(J) \neq \varnothing$.

(iv) J *is a dual potential basis iff* $\Lambda_1(J) \neq \varnothing$.

(v) J *is a potential basis iff* $\Gamma_1(J) \times \Lambda_1(J) \neq \varnothing$.

DEFINITION 2. The optimal situation set for a given potential basis J is defined by

$$S(J) = \{(\gamma, \lambda) | \gamma \in \Gamma_1(J),\ \lambda \in \Lambda_1(J)\}.$$

By Definition 2, whenever $(\gamma, \lambda) \in S(J)$, J is the optimal basis for model (1). Without confusion, $S(J)$ may simply be denoted by S. In order to use the MC^2-simplex method to search for a set of PGDs associated with certain ranges of (γ, λ), we need the following assumption (see [6]).

Assumption 1. (i) The number of products, k, in each PGD for model (1) should not exceed the number of resources under consideration, m.

(ii) The selected k products should be able to "optimize" model (1) under some possible ranges of (γ, λ).

From (i)–(v) of Statement 1, we see that if a PGD is a potential basis, then it satisfies both (i) and (ii) of Assumption 1; conversely, a PGD which satisfies both (i) and (ii) of Assumption 1 with some specific (γ, λ) can be represented by a potential basis. Thus, the method to search for a set of potential bases by the MC^2-simplex method can be readily used to locate a set of PGDs. For this reason, a potential basis of model (1) will be called a PGD for model (1). Thus, the products selected in the PGD are those we can potentially select as the optimal design.

DEFINITION 3. A PGD J is said to be a *non-trivial* PGD if J contains at least one product i.

From Definition 3, when a PGD J does not contain any products (that is, J is an initial potential basis) J is trivial. To select the optimal designs, we should consider the set of all non-trivial PGDs.

Let $\mathcal{P} = \{J_1, \ldots, J_r\}$ be the set of all non-trivial PGDs for model (1). Given a non-trivial PGD of $\mathcal{P}$, say J, if there is some $\lambda \notin \Lambda_1(J)$, then J has no optimal solution. In the following section, we focus on how to construct the corresponding FDCPs for a given non-trivial PGD of $\mathcal{P}$. With the FDCPs, we can flexibly adjust the number of selected products made, the contribution of the products, and the units of slack resources to overcome the non-optimal situation.

3. Constructing Flexible Dual Contingency Plans

To facilitate the discussion of constructing the FDCPs for each non-trivial PGD, we review some of duality concepts in MC^2 linear programming as follows (see [15] for details).

Let us call model (1) the *primal* (P) *model* of the design problem. Then we define its dual (D) model as

$$\begin{aligned} & \text{Min } u^t D\gamma \\ \text{s.t.} \quad & u^t[A, I_m] \geq \lambda^t[C, 0] \qquad (8) \\ & u^t \geq 0, \end{aligned}$$

where $u \in \mathbf{R}^m$ is an m-dimensional vector corresponding to the m primal constraints.

An economic interpretation of the (P) model for optimal linear designs may be maximizing the total contribution of selected products in PGDs by using resources $D\gamma$ "intelligently," while that of the (D) model may be minimizing the total "implicit value" of the resources $D\gamma$ consumed by producing the products in PGDs (see Hillier and Lieberman [5]).

Definition 4. Given a basis J for the (P) model, for each $\gamma \in \Gamma_1(J)$ the basic solution $x(J, \gamma) = B_J^{-1}D\gamma$ is a (P)-feasible solution; for each $\lambda \in \Lambda_1(J)$, $u^t(J, \lambda) = \lambda^t C_J B_J^{-1}$ is a (D)-feasible solution.

Given a non-trivial PGD J, $u^t(J, \lambda) = \lambda^t C_J B_J^{-1}$ may be viewed as the minimal unit implicit value of the resources $D\gamma$ or what is commonly known as the *shadow price* of $D\gamma$. Note that if $\Gamma_1(J) \neq \emptyset$, then $\Lambda_1(J) \neq \emptyset$ is the optimality condition for PGD J. That is, the feasible solution $x(J, \gamma) = B_J^{-1}D\gamma$, $\gamma \in \Gamma_1(J)$, is optimal if $\lambda \in \Lambda_1(J)$.

When a non-trivial PGD J is chosen as the optimal design, all other products $j \notin J$ are rejected or not produced. The product set of interest thus reduces to the products selected in J from N. However, when $\gamma \notin \Gamma_1(J)$, J is not feasible; and when $\lambda \notin \Lambda_1(J)$, J is not optimal. If some $\gamma \notin \Gamma_1(J)$, we can purchase external resources and build submodels of model (1) to construct the corresponding primal contingency plans for J. These contingency plans, including PGD J, can make the design feasible and optimal for various decision situations (see [6, 9, 11, 12]).

Recently, Shi [10] proposed a method to construct the RDCPs for a PGD. In this method for a given PGD, we first adjust the number of selected products made and the selected slack resource within the PGD, then adjust their contributions so that the PGD can satisfy the optimality condition for the changes of decision parameters. However, the flexibility of using the slack resources that are not in the PGD has not been discussed. To avoid the possible waste of these slack resources, we now introduce a method of constructing the corresponding FDCPs for a given non-trivial PGD, in which the number of selected products made, the contribution of the products, and the units of slack resources will be *flexibly* adjusted to meet the optimality. Keep in mind that rather than utilizing the *only* slack resources selected in the PGD J, we use *all* possible slack resources, including ones that are not in J, to build submodels of model (8) (or the (D) model) for locating all FDCPs.

Given a non-trivial PGD J, from (i) of Statement 1, $x(J, \gamma) = B_J^{-1}D\gamma$ means that resources (inputs) $D\gamma$ are converted into products (outputs) $x(J, \gamma)$ through the transformation B_J^{-1}. This can be rewritten as $B_J x(J) = D\gamma$. Note that if $x(J)$ contains some slack variable s_i, then the PGD J can only use s_i, not other slack variables in the production process. Thus, J has no flexibility to use other possible slack variables that are not in $x(J)$. In order to have the flexibility of using all possible slack variables, we shall use a flexible process $B_J x(J) \leq D\gamma$, rather than $B_J x(J) = D\gamma$.

We decompose the nonbasic variables $x(J')$ into $x^1(J')$ and $x^2(J')$, where $x^1(J')$ is called the *nonbasic slack variables* containing those slack variables that are not in J, and $x^2(J')$ is called the *nonbasic product variables* containing those product variables of N, but which are not in J (i.e., $N \backslash J$). We rearrange the index if necessary and decompose $[A, I_m]$ in (2) into B_J, R^1, and R^2, where R^1 is the submatrix of $[A, I_m]$ corresponding to the nonbasic slack variables $x^1(J')$ and R^2 is the submatrix of $[A, I_m]$ corresponding to the nonbasic product variables $x^2(J')$. Similarly, we decompose $[C, 0]$ in (3) into C_J, $C_{J'}^1$ and $C_{J'}^2$, where C_J, $C_{J'}^1$, and $C_{J'}^2$ are the coefficients of $[C, 0]$ for $x(J)$, $x^1(J')$, and $x^2(J')$, respectively. Note that $C_{J'}^1$ is a zero matrix since $x^1(J')$ are slack variables whose contribution coefficients are set equal to zero.

We add the nonbasic slack variables $x^1(J')$ into the left-hand side of the flexible process $B_J x(J) \leq D\gamma$ such that $B_J x(J) + R^1 x^1(J') = D\gamma$. The contribution in terms of the products selected in J is $\lambda^t C_J x(J)$. Thus, the subdesign model of using the PGD J will be

$$
\begin{aligned}
& \operatorname{Max}\ \lambda^t C_J x(J) \\
\text{s.t.} \quad & B_J x(J) + R^1 x^1(J') = D\gamma \qquad (9) \\
& x(J),\ x^1(J') \geq 0,
\end{aligned}
$$

where C_J, B_J, D, and R^1 are known and (γ, λ) are presumed.

To construct the FDCPs for the PGD J, we solve the dual model of model (9) which is defined by

$$\text{Min } u'D\gamma$$

$$\text{s.t.} \quad u'[B_J, R^1] \geq \lambda'[C_J, 0] \tag{10}$$

$$u' \text{ unrestricted.}$$

Note that model (10) is a submodel of model (8). We can get model (10) from model (8) by deleting columns of R^2 from model (8).

Let d be the number of nonbasic slack variables in $x^1(J')$. Then $n - d$ is the number of nonbasic slack variables in $x^2(J')$, since $x(J')$ has n nonbasic variables. Let $u^1(J')$ be the *dual surplus variables complementary to* $x^1(J')$ and $u(J)$ be the *dual surplus variables complementary to* $x(J)$. By putting $u^1(J')$ and $u(J)$ into model (10), it becomes

$$\text{Min } \gamma'D'u$$

$$\text{s.t.} \quad \begin{pmatrix} B'_J \\ (R^1)' \end{pmatrix} u - I_d u^1(J') - I_m u(J) = \begin{pmatrix} C'_J \\ 0 \end{pmatrix} \lambda$$

$$\text{u unrestricted; } u^1(J'),\ u(J) \geq 0.$$

The initial tableau of the above model is

u	$u^1(J')$	$u(J)$	RHS	
B'_J	0	$-I_m$	$C'_J\lambda$	
$(R^1)'$	$-I_d$	0	0	(11)
$\gamma'D'$	0	0	0	(12)

The dual basis of the table (11), (12) corresponding to the primal basis B_J can be written by

$$B_J^* = \begin{pmatrix} B'_J & 0 \\ (R^1)' & -I_d \end{pmatrix}$$

and the inverse matrix is

$$(B_J^*)^{-1} = \begin{pmatrix} (B'_J)^{-1} & 0 \\ (R^1)'(B'_J)^{-1} & -I_d \end{pmatrix}$$

(see Bazaraa *et al.* [1]).

Using $(B_J^*)^{-1}$, the table (11), (12) can be written as

u	$u^1(J')$	$u(J)$	RHS	
I_m	0	$-(B_J')^{-1}$	$(C_J B_J^{-1})'\lambda$	
0	I_d	$-(R^1)'(B_J')^{-1}$	$(C_J B_J^{-1} R^1)'\lambda$	(13)
0	0	$\gamma'(B_J^{-1} D)'$	$-\gamma'(C_J B_J^{-1} D)'\lambda$	(14)

where (13) $= (B_J^*)^{-1} \cdot$ (11), and (14) $= [-\gamma' D', 0] \cdot$ (13) + (12).

DEFINITION 5. Given a basis $K(J)$ for model (10), define its corresponding

(i) *primal parameter set* by

$$\Lambda_2(K(J)) = \{\lambda \geq 0 | \lambda'[C_J B_J^{-1}, C_J B_J^{-1} R^1] \geq 0\};$$

and the

(ii) *dual parameter set* by

$$\Gamma_2(K(J)) = \{\gamma \geq 0 | (B_J^{-1} D)\gamma \geq 0\}.$$

DEFINITION 6. Given a non-trivial PGD J, a basis $K(J)$ for model (10) is said to be a *trivial basis* corresponding to J if $K(J)$ contains all dual variables complementary to product variables associated with J.

For a given non-trivial PGD J, let $\{K_1(J), \ldots, K_h(J)\}$ be the set of all non-trivial potential bases (solutions) obtained from model (10) by using the MC^2-simplex method. We call $\{K_1(J), \ldots, K_h(J)\}$ the *FDCPs selected by model* (10) for the PGD J.

If for every (γ, λ) there is a non-trivial $K_i(J)$ such that $(\gamma, \lambda) \in \Gamma_2(K_i(J)) \times \Lambda_2(K_i(J))$, then we say that $\{K_1(J), \ldots, K_h(J)\}$ is the set of *all* FDCPs for the PGD J. If there is some $\lambda \notin \Lambda_2(K_i(J))$, $i = 1, \ldots, h$, then model (10) has no feasible solution. That is, model (9) has no optimal solution for the λ. To overcome this situation, let $\beta(J) = (\beta_1, \ldots, \beta_m)'$ be the vector of increments of the basic variables $x(J)$ and $\beta^1(J') = (\beta_{m+1}, \ldots, \beta_{m+d})'$ be the vector of increments of the slack variables $x^1(J')$. Let the unit contribution of these newly acquired $\beta(J)$ be $w(J)' = (w_1, \ldots, w_m)$, while $\beta^1(J')$ is worth $w^1(J')' = (w_{m+1}, \ldots, w_{m+d})$. (Note that for a slack variable s_j, we say that β_j is the additional amount of the jth resource saved, which is worth w_j per unit). When we add $w(J)'$ and $w^1(J')'$ into the right-hand side of model (10), the design system becomes

$$\text{Min } u^tD\gamma - w(J)^t\beta(J) - w^1(J')^t\beta^1(J')$$

$$\text{s.t.} \quad u^t[B_J, R^1] \geq \lambda^t[C_J, 0] + w(J)^t + w^1(J')^t \tag{15}$$

$$\text{with } u^t \text{ unrestricted, and } w(J)^t, w^1(J')^t \geq 0.$$

The initial tableau of model (15) can be written as

u	$u^1(J')$	$u(J)$	$w(J)$	$w^1(J')$	RHS	
B_J'	0	$-I_m$	$-I_m$	0	$\begin{pmatrix} C^t \\ 0 \end{pmatrix}\lambda$	(16)
$(R^1)^t$	$-I_d$	0	0	$-I_d$		
γ^tD^t	0	0	$-\gamma^t\mathbb{1}_q\beta(J)^t$	$-\gamma^t\mathbb{1}_q\beta^1(J')^t$	0	(17)

where $\mathbb{1}_p^t = (1, 1, \ldots, 1) \in \mathbf{R}^p$ and $\gamma^t\mathbb{1}_p = 1$ (normalization).

Using $(B_J^*)^{-1}$, the table (16), (17) can be further written as

u	$u^1(J')$	$u(J)$	$w(J)$	$w^1(J')$	RHS	
I_m	0	$-(B_J')^{-1}$	$-(B_J')^{-1}$	0	$(C_JB_J^{-1})^t\lambda$	
0	I_d	$-(R^1)^t(B_J')^{-1}$	$-(R^1)^t(B_J')^{-1}$	I_d	$(C_JB_J^{-1}R^1)^t\lambda$	(18)
0	0	$\gamma^t(B_J^{-1}D)^t$	$-\gamma^t[\beta(J)\mathbb{1}_q^t - B_J^{-1}D]^t$	$-\gamma^t\mathbb{1}_q\beta^1(J')^t$	$-\gamma^t(C_JB_J^{-1}D)^t\lambda$	(19)

where (18) = $(B_J^*)^{-1} \cdot$ (16) and (19) = $[-\gamma^tD^t, 0] \cdot$ (18) + (17).

Given a non-trivial PGD J, let $\Lambda_3(K(J))$ be the primal parameter set of a basis $K(J)$ for model (15) and $\Gamma_3(K(J))$ be the primal parameter set of a basis $K(J)$. We call the set of all non-trivial potential solutions obtained from model (15) by using the MC^2-simplex method the *FDCPs selected by model* (15). By adjusting $w(J)^t$ and $w^1(J')^t$, table (18), (19) can always provide feasible solutions. Thus, by using model (10) or/and model (15), we can always have the optimal FDCPs or all possible situations of (γ, λ).

4. An Augmented Dual Model and a Solution Procedure

In the previous section we have developed two dual models to construct the corresponding FDCPs for a given non-trivial PGD J. However, it is not effective to use this method to construct all FDCPs for each PGD of $\mathcal{P} = \{J_1, \ldots, J_r\}$ because there may be $2r$ related models which need to be built to complete the task! In order to explore an effective and systematic method to locate all FDCPs for each PGD of $\mathcal{P}$, we study the relationship between model (10), model (15), and the following augmented dual design model, which was developed by [10]:

$$\begin{aligned} & \text{Min } u^t D\gamma - w^t\beta \\ \text{s.t.} \quad & u^t[A, I_m] \geq \lambda^t[C, 0] + w^t \\ & u^t \text{ unrestricted, and } w^t \geq 0, \end{aligned} \tag{20}$$

where $w \in \mathbf{R}^{n+m}$ is a $(n+m)$-dimensional vector and $\beta \in \mathbf{R}^{n+m}$ is a $(n+m)$-dimensional parameter.

In model (20), $u^t D\gamma - w^t\beta$ are the net total implicit value of the resources $D\gamma$ after deducting the extra contribution of w from the total implicit value of the resources; and $u^t[A, I_m] \geq \lambda^t[C, 0] + w^t$ are constraints.

By putting surplus variables u_s into model (20), it becomes

$$\begin{aligned} & \text{Min } \gamma^t D^t u - \beta^t w \\ \text{s.t.} \quad & \begin{pmatrix} A^t \\ I_m \end{pmatrix} u - I_{n+m} u^s - I_{n+m} w = \begin{pmatrix} C^t \\ 0 \end{pmatrix} \lambda \\ & u \text{ unrestricted and } w, u_s \geq 0, \end{aligned} \tag{21}$$

where $u_s \in \mathbf{R}^{n+m}$ is a $(n+m)$-dimensional vector.

We can write the initial tableau of model (21) as

u	u_s^1	u_s^2	w^1	w^2	RHS
A^t I_m	$-I_n$ 0	0 $-I_m$	$-I_n$ 0	0 $-I_m$	$\begin{pmatrix} C^t \\ 0 \end{pmatrix}\lambda$
$\gamma^t D^t$	0	0	$-(\beta^1)^t$	$-(\beta^2)^t$	0

(22)

where u_s, w, and β^t are decomposed as (u_s^1, u_s^2), (w^1, w^2), and $((\beta^1)^t, (\beta^2)^t)$, respectively.

Given a non-trivial PGD J with its basis B_J and the primal basic variables $x(J)$, we partition (u, u_s^1, u_s^2) into $(u, u(J), u^1(J'), u^2(J'))$, where u are the *dual variables corresponding to the primal constraints*; $u(J)$ are the *dual surplus variables complementary to* $x(J)$; $u^1(J')$ are the *dual surplus variables complementary to* $x^1(J')$; and $u^2(J')$ are the *dual surplus variables complementary to* $x^2(J')$. Correspondingly, we also partition

(w^1, w^2) $(=w)$ into $(w(J), w^1(J'), w^2(J'))$ and $((\beta^1)^t, (\beta^2)^t)$ into $(\beta(J)^t, \beta^1(J')^t, \beta^2(J')^t)$, where $w(J)$ are the unit contributions of newly acquired units $\beta(J)$ of $x(J)$; $w^1(J')$ are the unit price of newly saved units $\beta^1(J')$ of $x^1(J')$; and $w^2(J')$ are the remaining $n - d$ components of w associated with $\beta^2(J')$ after excluding $w(J)$ and $w^1(J')$ from w. Then, table (22) can be rewritten as

u	$u^1(J')$	$u^2(J')$	$u(J)$	$w(J)$	$w^1(J')$	$w^2(J')$	RHS	
B_J^t	0	0	$-I_m$	$-I_m$	0	0	$C_J^t \lambda$	
$(R^1)^t$	$-I_d$	0	0	0	$-I_d$	0	$C_{J'}^{1t} \lambda$	(23)
$(R^2)^t$	0	$-I_{n-d}$	0	0	0	$-I_{n-d}$	$C_{J'}^{2t} \lambda$	
$\gamma^t D^t$	0	0	0	$-\gamma^t 1_q \beta(J)^t$	$-\gamma^t 1_q \beta^1(J')^t$	$-\gamma^t 1_q \beta^2(J')^t$	0	(24)

where C_J, $C_{J'}^1$ and $C_{J'}^2$ are the coefficients of $[C, 0]$ for $x(J)$, $x^1(J')$, and $x^2(J')$, respectively. Note that $C_{J'}^1$ is a zero matrix.

We know that the dual basis of table (23), (24) corresponding to the primal basis B_J can be represented by

$$B_J^{**} = \begin{pmatrix} B_J^t & 0 & 0 \\ (R^1)^t & -I_d & 0 \\ (R^2)^t & 0 & -I_{n-d} \end{pmatrix}$$

and the inverse matrix of B_J^{**} is

$$(B_J^{**})^{-1} = \begin{pmatrix} (B_J^t)^{-1} & 0 & 0 \\ (R^1)^t(B_J^t)^{-1} & -I_d & 0 \\ (R^2)^t(B_J^t)^{-1} & 0 & -I_{n-d} \end{pmatrix}$$

Using $(B_J^{**})^{-1}$, table (23), (24) can be written as

u	$u^1(J')$	$u^2(J')$	$u(J)$	$w(J)$	$w^1(J')$	$w^2(J')$	RHS	
I_m	0	0	$-(B_J^t)^{-1}$	$-(B_J^t)^{-1}$	0	0	$(C_J B_J^{-1})^t \lambda$	
0	I_d	0	$-(R^1)^t(B_J^t)^{-1}$	$-(R^1)^t(B_J^t)^{-1}$	I_d	0	$(C_J B_J^{-1} R^1)^t \lambda$	(25)
0	0	I_{n-d}	$-(R^2)^t(B_J^t)^{-1}$	$-(R^2)^t(B_J^t)^{-1}$	0	I_{n-d}	$(C_J B_J^{-1} R^2 - C_{J'}^2)^t \lambda$	
0	0	0	$\gamma^t(B_J^{-1}D)^t$	$-\gamma^t[\beta(J) 1_q^t - B_J^{-1}D]^t$	$-\gamma^t 1_q \beta^1(J')^t$	$-\gamma^t 1_q \beta^2(J')^t$	$-\gamma^t(C_J B_J^{-1} D)^t \lambda$	(26)

where (25) $= (B_J^{**})^{-1} \cdot (23)$ and (26) $= [-\gamma^t D^t, 0, 0] \cdot (25) + (24)$.

Comparing table (25), (26) with table (13), (14), we see that table (13), (14) can be easily obtained from table (25), (26) by dropping the columns of $u^2(J')$, $w(J)$, $w^1(J')$, and $w^2(J')$ and the third row block of (25). Furthermore, by using the same pivoting sequence each basic feasible solution (which may contain some u_j of u and $u^1(J')$ as the basic variables) that can be derived from table (13), (14) can also be derived from table (25), (26).

Let $K(J)$ be a basis with $u(K)$ and $u^1(K)$ as the corresponding basic variables in u and $u^1(J')$ respectively. Let $U(J)$ be another basis which also contains $u(K)$ and $u^1(K)$ as the basic variables. Then for the relationship of model (10) and model (20), we have

THEOREM 1. *If $K(J)$ is a basis for model* (10), *then $K(J)$ must be contained in a basis $U(J)$ for model* (20). *Conversely, if $U(J)$ is a basis for model* (20) *and $U(J)$ contains a basis $K(J)$ for model* (10), *then $U(J)$ can be reduced to $K(J)$ by deleting the variables of $u^2(J')$, $w(J)$, $w^1(J')$, and/or $w^2(J')$ from $U(J)$. Furthermore, the basic feasible solution of model* (10) *for a given λ, $(u(K, \lambda), u^1(K, \lambda))$, can be obtained by dropping the third row block of* (25) *from the MC^2-simplex tableau of model* (20).

Let B_K be the basis corresponding to $K(J)$ for model (10). Rearranging the index, we obtain

$$(u(K, \lambda), u^1(K, \lambda))^t = \lambda^t[C_J B_J^{-1}, C_J B_J^{-1} R^1](B_K^{-1})^t \tag{27}$$

Let $\Lambda_a(U(J))$ be the primal parameter sets for model (20) and $\Gamma_a(U(J, \beta))$ be the dual parameter sets for model (20). Then from Theorem 1 we have

COROLLARY 1. *For any basis $K(J)$ for model* (10) *and any basis $U(J)$ that contains $K(J)$ for model* (20),

(i) $\Lambda_a(U(J)) \subseteq \Lambda_2(K(J))$;

and

(ii) $\Gamma_a(U(J, \beta)) \subseteq \Gamma_2(K(J))$, *where* $\beta \in \mathbf{R}^m$.

In order to see the relationship of model (15) and model (20), let $K^*(J)$ be a basis with $u(K^*)$, $u^1(K^*)$, $w(K^*)$, and $w^1(K^*)$ as the corresponding basic variables in u, $u^1(J')$, $w(J)$, and $w^1(J')$ respectively. Let $U^*(J)$ be another basis which also contains $u(K^*)$, $u^1(K^*)$, $w(K^*)$, and $w^1(K^*)$ as the basic variables. Then, we have

THEOREM 2. *If $K^*(J)$ is a basis for model* (15), *then $K^*(J)$ must be contained in a basis $U^*(J)$ for model* (20). *Conversely, if $U^*(J)$ is a basis for model* (20) *and $U^*(J)$ contains a basis $K^*(J)$ for model* (15), *then*

$U^*(J)$ can be reduced to $K^*(J)$ by deleting the variables of $u^2(J')$ and/or $w^2(J')$ from $U^*(J)$. Furthermore, the basic feasible solution of model (15) *for a given λ, $(u(K^*, \lambda), u^1(K^*, \lambda), w(K^*, \lambda), w^1(K^*, \lambda))$, can be obtained by dropping the third row block of* (25) *from the MC^2-simplex tableau of model* (20).

Let B_{K^*} be the basis corresponding to $K^*(J)$ for model (15). Similarly to (27), we have

$$\begin{aligned}(u(K^*, \lambda), u^1(K^*, \lambda), w(K^*, \lambda), w^1(K^*, \lambda))^t \\ = \lambda^t[C_J B_J^{-1}, C_J B_J^{-1} R^1](B_{K^*}^{-1})^t \end{aligned} \tag{28}$$

A corollary of Theorem 2 can be

COROLLARY 2. *For any basis $K^*(J)$ for model* (15) *and any basis $U^*(J)$ that contains $K^*(J)$ for model* (20),

(i) $\Lambda_a(U^*(J)) \subseteq \Lambda_3(K^*(J))$;

and

(ii) $\Gamma_a(U^*(J, \beta)) \subseteq \Gamma_3(K^*(J, \beta))$, *where* $\beta \in \mathbf{R}^m$.

Remark 1. The (D) model (model (8)) is also a submodel of model (20) since the simplex table of the (D) model can be obtained by dropping the columns of $w(J)$, $w^1(J')$, and $w^2(J')$ from table (25), (26). Using a similar analysis, we can study the relationship between model (10) and model (15). We shall not stop here to do so.

Utilizing the above theoretical results, we propose a solution procedure of effectively and systematically locating all FDCPs for each PGD of $\mathcal{P}$ as follows:

Procedure 1

Step 1. Find all non-trivial PGDs (potential solutions) for the original design model (1). Denote such a solution set by $\mathcal{P} = \{J_1, \ldots, J_r\}$. Without confusion, J will represent such a basis and the corresponding basic variables are denoted by $x(J)$. Note that $x(J)$ can contain slack variables s.

Step 2. Find all possible bases for the augmented dual model (20). Denote such bases by $\mathcal{U} = \{U_1, \ldots, U_z\}$. Without confusion, U will represent such a basis, and the corresponding basic variables in u, $u^1(J')$, $w(J)$, and $w^1(J')$ will be denoted by $u(U)$, $u^1(U)$, $w(U)$, and $w^1(U)$ respectively.

Step 3. For each PGD J of Step 1, identify its corresponding dual bases of model (10) from $\mathcal{U}$. Using Theorem 1, this can be done by dropping the columns of $u^2(J')$, $w(J)$, $w^1(J')$, and $w^2(J')$ from the table (25),

(26) associated with the PGD J and the third-row block of (25). Denote such reduced bases of $\mathcal{U}$ by $\mathcal{U}(J) = \{U_1(J), \ldots, U_k(J)\}$. Then we use Definition 5 to identify the set of non-trivial potential bases for model (10) from $\mathcal{U}(J)$, denoted by $\mathcal{U}^*(J) = \{U_1(J), \ldots, U_h(J)\}$. Note that for each $U(J)$ of $\mathcal{U}^*(J)$, whenever $(\gamma, \lambda) \in \Gamma_2(U(J)) \times \Lambda_2(U(J))$, $(u(U, \lambda), u^1(U, \lambda))$ of (27) will be a FDCP selected by model (10) for the PGD J. If for each (γ, λ) there is $U_i(J)$ of $\mathcal{U}^*(J)$ such that $(\gamma, \lambda) \in \Gamma_2(U_i(J)) \times \Lambda_2(U_i(J))$, $i = 1, \ldots, h$, then $\mathcal{U}^*(J)$ is the set of all FDCPs for J and we go to Step 5; otherwise, we go to Step 4.

Step 4. For each PGD J of Step 1, with a given β^0 value, identify its corresponding dual bases of model (15) from $\mathcal{U}$. Using Theorem 2, we drop the columns of $u^2(J')$ and $w^2(J')$ from table (25), (26) and the third row block of (25). The resulting subset of $\mathcal{U}$ is the basis of model (15). Then from this subset, we find the set of non-trivial potential bases for model (15), denoted by $\mathcal{U}^{**}(J)$. For each element $U(J)$ of $\mathcal{U}^{**}(J)$, when $(\gamma, \lambda) \in \Gamma_3(U(J, \beta^0)) \times \Lambda_3(U(J))$, $(u(U, \lambda), u^1(U, \lambda), w(U, \lambda), w^1(U, \lambda))$ of (28) is a FDCP selected by model (15) for the PGD J. When a proper β^0 value is chosen, $\mathcal{U}^{**}(J)$ must be the set of all FDCPs for the PGD J.

Step 5. Evaluate each PGD J and its corresponding FDCPs in terms of a *criterion* preferred by the decision maker. A number of known criteria of decision making under uncertainty, such as maximizing expected payoff, minimizing the variance of the payoff, maxi-min payoff, maximizing the probability of achieving a targeted payoff, stochastic dominance, probability dominance, and mean-variance dominance, can be used to solve the problem (see [9]). The final optimal designs (systems) are selected based on the evaluation.

In the next section, an example is used to illustrate Procedure 1 for selecting optimal linear designs and constructing their corresponding FDCPs.

5. An Illustrative Example

Consider the following optimal linear design problem with two objectives and two resource availability levels:

$$\text{Max } (\lambda_1, \lambda_2)\begin{pmatrix} -4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{s.t.} \qquad \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 40 & 30 \\ 10 & 15 \end{pmatrix}\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$$

$$x_i \geq 0, \qquad i = 1, 2.$$

By using Procedure 1, we have the following.

Step 1. Let $s_j \geq 0$, $j = 1, 2$, be slack variables. Then by using the computer program of Chien *et al.* [3] we solve the above problem (model (1)) and obtain the set of non-trivial PGDs, denoted by $\mathcal{P} = \{J_1, J_2, J_3\}$ as in Table I.

From Table I, we see that J_1 has (x_1, s_1) as the basic variables. If J_1 is chosen, then product {2} is the one that we select for commitment and s_1 is the amount of slack resources left over from constraint 1. When γ_1 and λ_1 are given, γ_2 and λ_2 are uniquely specified due to $\gamma_1 + \gamma_2 = 1$ and $\lambda_1 + \lambda_2 = 1$. Thus J_1 is a PGD whenever $3/5 \leq \gamma_1 \leq 1$ and $0 \leq \lambda_1 \leq 1/3$. However, if $0 \leq \gamma_1 < 3/5$, J_1 becomes infeasible, and if $1/3 < \lambda_1 \leq 1$, J_1 is not optimal. We must construct the corresponding contingency plans for J_1 when $0 \leq \gamma_1 < 3/5$ and/or $1/3 < \lambda_1 \leq 1$. If PGD J_2 is chosen, when $0 \leq \gamma_1 \leq 3/5$ and $0 \leq \lambda_1 \leq 1/6$, products {1, 2} are ones to be undertaken. PGD J_3 can be similarly explained. When $1/3 < \lambda_1 \leq 1$, the given design model has no optimal solutions (see the black region in Fig. 1).

By Definition 2, we get the following optimal situation sets:

$$S_1 = \{(\gamma_1, \lambda_1) | 3/5 \leq \gamma_1 \leq 1,\ 0 \leq \lambda_1 \leq 1/3\},$$

$$S_2 = \{(\gamma_1, \lambda_1) | 0 \leq \gamma_1 \leq 3/5,\ 0 \leq \lambda_1 \leq 1/6\},$$

and

$$S_3 = \{(\gamma_1, \lambda_1) | 0 \leq \gamma_1 \leq 3/5,\ 1/6 \leq \lambda_1 \leq 1/3\},$$

where (γ, λ) defined in S of Definition 2 reduces to (γ_1, λ_1).

Figure 1 shows how $\{S_1, S_2, S_3\}$ graphically decomposes the space (γ_1, λ_1).

Step 2. Let β_i be the increments of the ith product and its unit contribution be w_i, $i = 1, 2, 3, 4$. Then the corresponding augmented dual model (model (20)) is

TABLE I

Potentially Good Designs for Model (1)

Potentially good designs	Basic variables	$\Gamma_1(J_i)$	$\Lambda_1(J_i)$
J_1	(x_2, s_1)	$\frac{3}{5} \leq \gamma_1 \leq 1$	$0 \leq \lambda_1 \leq \frac{1}{3}$
J_2	(x_1, x_2)	$0 \leq \gamma_1 \leq \frac{3}{5}$	$0 \leq \lambda_1 \leq \frac{1}{6}$
J_3	(x_2, s_2)	$0 \leq \gamma_1 \leq \frac{3}{5}$	$\frac{1}{6} \leq \lambda_1 \leq \frac{1}{3}$

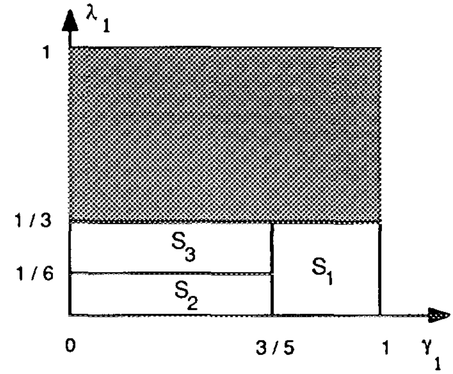

FIG. 1. Potentially good designs.

$$\text{Min } (\gamma_1, \gamma_2)\begin{pmatrix} 40 & 10 \\ 30 & 15 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - (\beta_1, \beta_2, \beta_3, \beta_4)\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}$$

s.t.

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} \geq \begin{pmatrix} -4 & 1 \\ -2 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

u_j, $j = 1, 2$, are unrestricted; and $w_i \geq 0$, $i = 1, 2, 3, 4$.

Let $u_j \geq 0, j = 3, 4, 5, 6$, be surplus variables. Then using the computer program [3], we obtain 89 bases, denoted by $\mathcal{U} = \{U_1, \ldots, U_{89}\}$.

Step 3. For each J_i of $\mathcal{P}$, $i = 1, 2, 3$, by using Theorem 1 we first identify their corresponding dual bases from $\mathcal{U}$ by dropping the columns of $u^2(J_i')$, $w(J_i)$, $w^1(J_i)$, $w^1(J_i')$, and $w^2(J_i')$ from the table (25), (26) associated with the PGD J_i and the third-row block of (25). Then, using Definition 5, we find sets of FDCPs selected by model (10) for J_i, denoted by $\mathcal{U}^*(J_i)$ as in Tables II–IV, respectively. Note that in Tables II–IV $\Lambda_2(U_j(J_i))$ is the primal parameter set of $U_j(J_i)$; $\Gamma_2(U_j(J_i))$ is the dual parameter set of $U_j(J_i)$; and $V(U_j|J_i)$ is defined as the payoff of $U_j(J_i)$. The graphical representation of Tables II–IV is Figs. 2–4. For simplicity, we use U_i to indicate its S_i in Figs. 2–4.

For the PGD J_1, we have that $x(J_1) = (x_2, s_1)$, $x^1(J_1') = (s_2)$, and $x^2(J_1') = (x_1)$. The partition of corresponding complementary dual variables is

TABLE II

Flexible Dual Contingency Plans Selected by Model (10) for J_1

$U_j(J_1)$	$\Lambda_2(U_j(J_1))$	$\Lambda_2(U_j(J_1))$	Payoff $V(U_j \mid J_1)$
$u(U_{21}) = (u_1, u_2, u_5)$	$0 \le \lambda_1 \le \frac{1}{3}$	$0 \le \gamma_1 \le \frac{3}{5}$	$(\gamma_1, \gamma_2)\begin{pmatrix} -26.7 & 13.3 \\ -20 & 10 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{50}) = (u_1, u_5, u_6)$	$0 \le \lambda_1 \le \frac{1}{3}$	$0 \le \gamma_1 \le \frac{3}{5}$	$(\gamma_1, \gamma_2)\begin{pmatrix} -26.7 & 13.3 \\ -20 & 10 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{20}) = (u_1, u_2, u_6)$	$0 \le \lambda_1 \le \frac{1}{3}$	$\frac{3}{5} \le \gamma_1 \le 1$	$(\gamma_1, \gamma_2)\begin{pmatrix} -20 & 10 \\ -30 & 15 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{52}) = (u_2, u_5, u_6)$	$0 \le \lambda_1 \le \frac{1}{3}$	$\frac{3}{5} \le \gamma_1 \le 1$	$(\gamma_1, \gamma_2)\begin{pmatrix} -20 & 10 \\ -30 & 15 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$

$u(J_1) = (u_4, u_5)$, $u^1(J_1') = (u_6)$, and $u^2(J_1') = (u_3)$, respectively. The dual variables $u = (u_1, u_2)$ are complementary to the two primal constraints. Then the meaning of Table II is as follows:

If PGD J_1 is chosen, (i) whenever $0 \le \lambda_1 \le \frac{1}{3}$ and $0 \le \gamma_1 \le \frac{3}{5}$, then the dual basis U_{21} or U_{50} can be used. When U_{21} is used, by the complementary slackness conditions in linear programming (see [1]), we see that $x_2 \ge 0$ since the corresponding complementary variable $u_4 = 0$ (i.e., u_4 is not in U_{21}); $s_1 = 0$ since the corresponding complementary variable $u_5 > 0$ (i.e., u_5 is in U_{21}); and $s_2 \ge 0$ since the corresponding complementary variable

TABLE III

Flexible Dual Contingency Plans Selected by Model (10) for J_2

$U_j(J_2)$	$\Lambda_2(U_j(J_2))$	$\Gamma_2(U_j(J_2))$	Payoff $V(U_j \mid J_2)$
$u(U_3) = (u_1, u_2, u_5, u_6)$	$0 \le \lambda_1 \le \frac{1}{6}$	$0 \le \gamma_1 \le \frac{3}{5}$	$(\gamma_1, \gamma_2)\begin{pmatrix} -20 & 12 \\ -30 & 12 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{52}) = (u_2, u_3, u_5, u_6)$	$0 \le \lambda_1 \le \frac{1}{3}$	$\frac{3}{5} \le \gamma_1 \le 1$	$(\gamma_1, \gamma_2)\begin{pmatrix} -20 & 10 \\ -30 & 15 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{20}) = (u_1, u_2, u_3, u_6)$	$0 \le \lambda_1 \le \frac{1}{3}$	$\frac{3}{5} \le \gamma_1 \le 1$	$(\gamma_1, \gamma_2)\begin{pmatrix} -20 & 10 \\ -30 & 15 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{21}) = (u_1, u_2, u_3, u_5)$	$\frac{1}{6} \le \lambda_1 \le \frac{1}{3}$	$0 \le \gamma_1 \le \frac{3}{5}$	$(\gamma_1, \gamma_2)\begin{pmatrix} -26.7 & 13.3 \\ -20 & 10 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{50}) = (u_1, u_3, u_5, u_6)$	$\frac{1}{6} \le \lambda_1 \le \frac{1}{3}$	$0 \le \gamma_1 \le \frac{3}{5}$	$(\gamma_1, \gamma_2)\begin{pmatrix} -26.7 & 13.3 \\ -20 & 10 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$

TABLE IV

Flexible Dual Contingency Plans Selected by Model (10) for J_3

$U_j(J_3)$	$\Lambda_2(U_j(J_3))$	$\Gamma_2(U_j(J_3))$	Payoff $V(U_j \mid J_3)$
$u(U_{21}) = (u_1, u_2, u_5)$	$0 \le \lambda_1 \le \frac{1}{3}$	$0 \le \gamma_1 \le \frac{3}{5}$	$(\gamma_1, \gamma_2)\begin{pmatrix} -26.7 & 13.3 \\ -20 & 10 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{50}) = (u_1, u_5, u_6)$	$0 \le \lambda_1 \le \frac{1}{3}$	$0 \le \gamma_1 \le \frac{3}{5}$	$(\gamma_1, \gamma_2)\begin{pmatrix} -26.7 & 13.3 \\ -20 & 10 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{20}) = (u_1, u_2, u_6)$	$0 \le \lambda_1 \le \frac{1}{3}$	$\frac{3}{5} \le \gamma_1 \le 1$	$(\gamma_1, \gamma_2)\begin{pmatrix} -20 & 10 \\ -30 & 15 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{52}) = (u_2, u_5, u_6)$	$0 \le \lambda_1 \le \frac{1}{3}$	$\frac{3}{5} \le \gamma_1 \le 1$	$(\gamma_1, \gamma_2)\begin{pmatrix} -20 & 10 \\ -30 & 15 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$

$u_6 = 0$ (i.e., u_6 is not in U_{21}). This means that we can produce product {2} and the slack resource s_2 is left over. Similarly, when U_{50} is used, we can produce product {2} and no slack resource is left over. (ii) Whenever $0 \le \lambda_1 \le \frac{1}{3}$ and $\frac{3}{5} \le \gamma_1 \le 1$, we can use either U_{20} or U_{52}. When U_{21} is used, we can produce product {2} and the slack resource s_1 is left over. When U_{52} is used, we can produce product {2} and no slack resource is left over. Therefore, we have four alterative FDCP sets for J_1: $\{U_{21}, U_{20}\}$, $\{U_{21}, U_{52}\}$, $\{U_{50}, U_{20}\}$, and $\{U_{50}, U_{52}\}$.

Tables III, IV can be similarly explained. However, because when $\frac{1}{3} < \lambda_1 \le 1$ model (10) for the PGD J_i has no optimal solution, we need to go to Step 4 to use model (15) for J_i to locate all FDCPs for J_i.

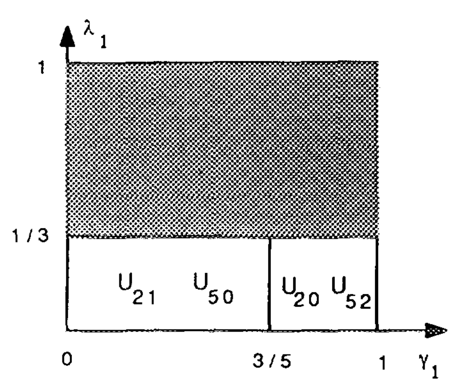

FIG. 2. Flexible dual contingency plans for J_1.

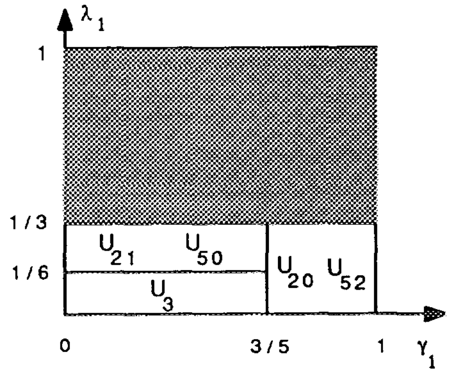

FIG. 3. Flexible dual contingency plans for J_2.

Step 4. For each PGD J_i of $\mathcal{P}$, $i = 1, 2, 3$, we use Theorem 2 to identify the sets of FDCPs selected by model (15) for J_i. This can be easily done by dropping the columns of $u^2(J_i^l)$ and $w^2(J_i^l)$ from the table (25), (26) and the third-row block of (25). Let $\Lambda_3(U_j(J_i))$ be the primal parameter set of $U_j(J_i)$, and let $\Gamma_3(U_j(J_i, \beta))$, which is a function of β, be the dual parameter set of $U_j(J_i)$. For illustrative purposes, let $\beta_i^0 = 2$, $i = 1, 2, 3, 4$. We obtain all FDCPs as in Tables V–VII or as shown graphically in Figs. 5–7. If J_1 is chosen for commitment, then Table V can be explained as follows:

(i) When $\frac{1}{3} \leq \lambda_1 \leq 1$ and $0 \leq \gamma_1 \leq 1$ occur, we use a basis U_{13}, U_{40}, or U_{83} to produce product {2}. When U_{13} is used, x_2 has increased two units (i.e., $\beta_2 = 2$) and the corresponding increment of the unit contribution is $w_2 =$

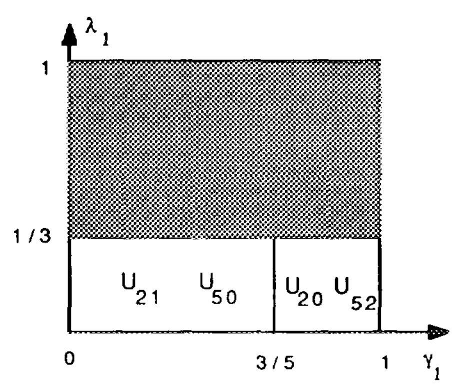

FIG. 4. Flexible dual contingency plans for J_3.

TABLE V

Flexible Dual Contingency Plans Selected by Model (15) for J_1 with Each $\beta_i^0 = 2$

$U_j(J_1)$	$\Lambda_3(U_j(J_1))$	$\Gamma_3(U_j(J_1))$	Payoff $V(U_j \mid J_1)$
$u(U_{13}) = (u_1, u_2, w_2)$	$\frac{1}{3} \le \lambda_1 \le 1$	$0 \le \gamma_1 \le 1$	$(\gamma_1, \gamma_2)\begin{pmatrix} -4 & 2 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{40}) = (u_1, w_2, w_4)$	$\frac{1}{3} \le \lambda_1 \le 1$	$0 \le \gamma_1 \le 1$	$(\gamma_1, \gamma_2)\begin{pmatrix} -4 & 2 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{83}) = (w_2, w_3, w_4)$	$\frac{1}{3} \le \lambda_1 \le 1$	$0 \le \gamma_1 \le 1$	$(\gamma_1, \gamma_2)\begin{pmatrix} -4 & 2 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{17}) = (u_1, u_2, w_3)$	$0 \le \lambda_1 \le \frac{1}{3}$	$0 \le \gamma_1 \le \frac{11}{25}$	$(\gamma_1, \gamma_2)\begin{pmatrix} -25.3 & 12.7 \\ -18.7 & 9.3 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{46}) = (u_1, w_3, w_4)$	$0 \le \lambda_1 \le \frac{1}{3}$	$0 \le \gamma_1 \le \frac{11}{25}$	$(\gamma_1, \gamma_2)\begin{pmatrix} -25.3 & 12.7 \\ -18.7 & 9.3 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{23}) = (u_1, u_2, w_4)$	$0 \le \lambda_1 \le \frac{1}{3}$	$\frac{11}{25} \le \gamma_1 \le 1$	$(\gamma_1, \gamma_2)\begin{pmatrix} -16 & 8 \\ -26 & 13 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{72}) = (u_2, w_3, w_4)$	$0 \le \lambda_1 \le \frac{1}{3}$	$\frac{11}{25} \le \gamma_1 \le 1$	$(\gamma_1, \gamma_2)\begin{pmatrix} -16 & 8 \\ -26 & 13 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$

$2\lambda_1 - \lambda_2$ from the related MC^2- simplex tableau of U_{13} in model (15). When U_{40} is used, x_2 has increased two units (i.e., $\beta_2 = 2$) and the corresponding increment of the unit contribution is $w_2 = 2\lambda_1 - \lambda_2$. The slack resource s_2 can be saved two more units (i.e., $\beta_4 = 2$), which are worth nothing because $w_4 = 0$ from the related MC^2-simplex tableau of U_{40}. When U_{83} is

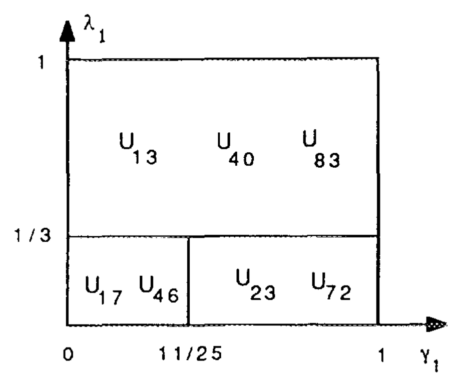

FIG. 5. Flexible dual contingency plans for J_1.

TABLE VI

Flexible Dual Contingency Plans Selected by Model (15) for J_2 with Each $\beta_i^0 = 2$

$U_j(J_2)$	$\Lambda_3(U_j(J_2))$	$\Gamma_3(U_j(J_2))$	Payoff $V(U_j \mid J_2)$
$u(U_5) = (u_1, u_2, w_1, w_2)$	$\frac{1}{3} \le \lambda_1 \le 1$	$0 \le \gamma_1 \le 1$	$(\gamma_1, \gamma_2)\begin{pmatrix} -12 & 4 \\ -12 & 4 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{31}) = (u_1, w_1, w_2, w_4)$	$\frac{1}{3} \le \lambda_1 \le 1$	$0 \le \gamma_1 \le 1$	$(\gamma_1, \gamma_2)\begin{pmatrix} -12 & 4 \\ -12 & 4 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{64}) = (u_2, w_1, w_2, w_3)$	$\frac{1}{3} \le \lambda_1 \le 1$	$0 \le \gamma_1 \le 1$	$(\gamma_1, \gamma_2)\begin{pmatrix} -12 & 4 \\ -12 & 4 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{65}) = (u_2, w_1, w_2, w_5)$	$\frac{1}{3} \le \lambda_1 \le 1$	$0 \le \gamma_1 \le 1$	$(\gamma_1, \gamma_2)\begin{pmatrix} -12 & 4 \\ -12 & 4 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_6) = (u_1, u_2, w_1, w_3)$	$\frac{1}{6} \le \lambda_1 \le \frac{1}{3}$	$0 \le \gamma_1 \le \frac{1}{25}$	$(\gamma_1, \gamma_2)\begin{pmatrix} -32 & 14 \\ -25.3 & 10.7 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{32}) = (u_1, w_1, w_3, w_4)$	$\frac{1}{6} \le \lambda_1 \le \frac{1}{3}$	$0 \le \gamma_1 \le \frac{1}{25}$	$(\gamma_1, \gamma_2)\begin{pmatrix} -32 & 14 \\ -25.3 & 10.7 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{16}) = (u_1, u_2, w_3, w_4)$	$0 \le \lambda_1 \le \frac{1}{6}$	$0 \le \gamma_1 \le \frac{1}{25}$	$\gamma_1, \gamma_2)\begin{pmatrix} -16 & 10.8 \\ -26 & 10.8 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_7) = (u_1, u_2, w_1, w_4)$	$0 \le \lambda_1 \le \frac{1}{3}$	$\frac{1}{25} \le \gamma_1 \le 1$	$(\gamma_1, \gamma_2)\begin{pmatrix} -16 & 6 \\ -26 & 11 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{63}) = (u_2, w_1, w_3, w_4)$	$0 \le \lambda_1 \le \frac{1}{3}$	$\frac{1}{25} \le \gamma_1 \le 1$	$(\gamma_1, \gamma_2)\begin{pmatrix} -16 & 6 \\ -26 & 11 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$

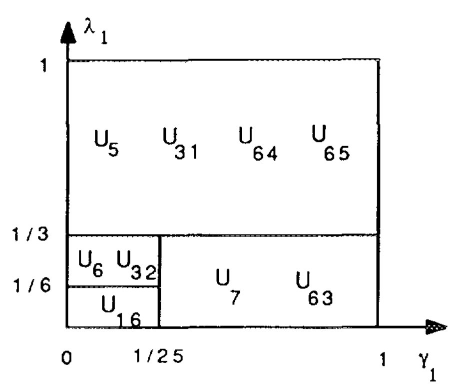

FIG. 6. Flexible dual contingency plans for J_2.

TABLE VII

Flexible Dual Contingency Plans Selected by Model (15) for J_3 with Each $\beta_i^0 = 2$

$U_j(J_1)$	$\Lambda_3(U_j(J_1))$	$\Gamma_3(U_j(J_1))$	Payoff $V(U_j \mid J_1)$
$u(U_{13}) = (u_1, u_2, w_2)$	$\frac{1}{3} \le \lambda_1 \le 1$	$0 \le \gamma_1 \le 1$	$(\gamma_1, \gamma_2)\begin{pmatrix} -4 & 2 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{40}) = (u_1, w_2, w_4)$	$\frac{1}{3} \le \lambda_1 \le 1$	$0 \le \gamma_1 \le 1$	$(\gamma_1, \gamma_2)\begin{pmatrix} -4 & 2 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{83}) = (w_2, w_3, w_4)$	$\frac{1}{3} \le \lambda_1 \le 1$	$0 \le \gamma_1 \le 1$	$(\gamma_1, \gamma_2)\begin{pmatrix} -4 & 2 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{17}) = (u_1, u_2, w_3)$	$0 \le \lambda_1 \le \frac{1}{3}$	$0 \le \gamma_1 \le \frac{11}{25}$	$(\gamma_1, \gamma_2)\begin{pmatrix} -25.3 & 12.7 \\ -18.7 & 9.3 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{46}) = (u_1, w_3, w_4)$	$0 \le \lambda_1 \le \frac{1}{3}$	$0 \le \gamma_1 \le \frac{11}{25}$	$(\gamma_1, \gamma_2)\begin{pmatrix} -25.3 & 12.7 \\ -18.7 & 9.3 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{23}) = (u_1, u_2, w_4)$	$0 \le \lambda_1 \le \frac{1}{3}$	$\frac{11}{25} \le \gamma_1 \le 1$	$(\gamma_1, \gamma_2)\begin{pmatrix} -16 & 8 \\ -26 & 13 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$
$u(U_{72}) = (u_2, w_3, w_4)$	$0 \le \lambda_1 \le \frac{1}{3}$	$\frac{11}{25} \le \gamma_1 \le 1$	$(\gamma_1, \gamma_2)\begin{pmatrix} -16 & 8 \\ -26 & 13 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$

used, x_2 has increased two units (i.e., $\beta_2 = 2$) and the corresponding increment of the unit contribution is $w_2 = 2\lambda_1 - \lambda_2$; and both slack resources s_1 and s_2 can save two more units (i.e., $\beta_3 = 2$, $\beta_4 = 2$), respectively. However, the prices of β_3 and β_4 are zero because $w_2 = w_4 = 0$ from the related MC^2-simplex tableau of U_{83}.

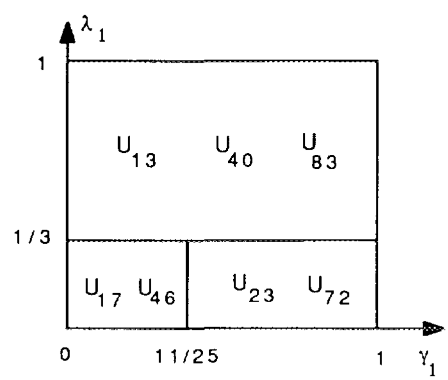

FIG. 7. Flexible dual contingency plans for J_3.

(ii) When $0 \leq \lambda_1 \leq \frac{1}{3}$ and $0 \leq \gamma_1 \leq \frac{11}{25}$ occur, we use either U_{17} or U_{46} to produce product {2}. When U_{17} is used, x_2 has no increment; but the slack resource s_1 can be saved two more units (i.e., $\beta_3 = 2$), which are worth $w_3 = -0.67\ \lambda_1 + 0.33\ \lambda_2$ from the related MC^2-simplex tableau of U_{17}. When U_{46} is used, x_2 has no increment; the slack resource s_1 can be saved two more units (i.e., $\beta_3 = 2$), which are worth $w_3 = -0.67\ \lambda_1 + 0.33\ \lambda_2$; and the slack resource s_2 can be saved two more units (i.e., $\beta_4 = 2$), which are worth nothing because $w_4 = 0$ from the related MC^2-simplex tableau of U_{46}.

(iii) When $0 \leq \lambda_1 \leq \frac{1}{3}$ and $\frac{11}{25} \leq \gamma_1 \leq 1$ occur, we use either U_{23} or U_{72} to produce product {2}. When U_{23} is used, x_2 has no increment, but the slack resource s_2 can be saved two more units (i.e., $\beta_4 = 2$) which are worth $w_4 = -2\lambda_1 + \lambda_2$ from the related MC^2-simplex tableau of U_{23}. When U_{72} is used, x_2 has no increment. The slack resource s_2 can be saved two more units (i.e., $\beta_4 = 2$), which are worth $w_4 = -2\lambda_1 + \lambda_2$, and the slack resource s_1 can be saved two more units (i.e., $\beta_3 = 2$), which are worth nothing because $w_3 = 0$ from the related MC^2-simplex tableau of U_{72}.

The set of *all* FDCPs for the PGD J_1 is $\{U_{13}, U_{40}, U_{83}, U_{17}, U_{46}, U_{23}, U_{72}\}$. For the space (γ_1, λ_1), this set produces twelve alternative FDCP sets for J_1: $\{U_{13}, U_{17}, U_{23}\}$, $\{U_{13}, U_{17}, U_{72}\}$, $\{U_{13}, U_{46}, U_{23}\}$, $\{U_{13}, U_{46}, U_{72}\}$, $\{U_{40}, U_{17}, U_{23}\}$, $\{U_{40}, U_{17}, U_{72}\}$, $\{U_{40}, U_{46}, U_{23}\}$, $\{U_{40}, U_{46}, U_{72}\}$, $\{U_{83}, U_{17}, U_{23}\}$, $\{U_{83}, U_{17}, U_{72}\}$, $\{U_{83}, U_{46}, U_{23}\}$, and $\{U_{83}, U_{46}, U_{72}\}$. Similarly to Table V, Tables VI, VII can be explained.

Step 5. Suppose that the decision maker would like to use the *maximizing expected payoff* as the criterion to evaluate each J_i of $\mathcal{P}$ and its corresponding FDCPs.

Assume that γ_1 is independent of λ_1, and let (γ_1, λ_1) have the *joint and uniform* probability distribution $F(\gamma_1, \lambda_1)$. Given an FDCP set $\{U_{13}, U_{17}, U_{23}\}$ for J_1, we can find the expected payoff $\mathbf{EV}(J_1)$ by computing

$$\mathbf{EV}(J_1) = \int_{S_1} V(U_{13}|J_1)\, dF(\gamma_1, \lambda_1) + \int_{S_2} V(U_{17}|J_1)\, dF(\gamma_1, \lambda_1) + \int_{S_3} V(U_{23}|J_1)\, dF(\gamma_1, \lambda_1),$$

where

$$V(U_{13}|J_1) = (\gamma_1, \gamma_2)\begin{pmatrix} -4 & 2 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix},$$

$$V(U_{17}|J_1) = (\gamma_1, \gamma_2)\begin{pmatrix} -25.3 & 12.7 \\ -18.7 & 9.3 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix},$$

and

$$V(U_{23}|J_1) = (\gamma_1, \gamma_2)\begin{pmatrix} -16 & 8 \\ -26 & 13 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix},$$

$F(\gamma_1, \lambda_1)$ is the cumulative uniform probability distribution function of (γ_1, λ_1) as assumed, and

$$S_1 = \{(\gamma_1, \lambda_1)|\ 0 \leq \gamma_1 \leq 1 \text{ and } \tfrac{1}{3} \leq \lambda_1 \leq 1\},$$
$$S_2 = \{(\gamma_1, \lambda_1)|\ 0 \leq \gamma_1 \leq \tfrac{11}{25} \text{ and } 0 \leq \lambda_1 \leq \tfrac{1}{3}\},$$

and

$$S_3 = \{(\gamma_1, \lambda_1)|\ \tfrac{11}{25} \leq \gamma_1 \leq 1 \text{ and } 0 \leq \lambda_1 \leq \tfrac{1}{3}\}$$

(see Table V).

We get $\mathbf{EV}(J_1) = -12.09$. If we use other FDCP sets for J_1, the result will be same.

Similarly, we can get $\mathbf{EV}(J_2) = 37.06$ and $\mathbf{EV}(J_3) = -12.09$.

Therefore, in view of *maximizing expected payoff*, J_2 should be the final optimal design (system) preferred by the decision maker, where the set of all FDCPs for J_2 are shown in Table VI or Fig. 6.

6. Conclusions

Given a linear design problem with multiple objectives and multiple resource availability levels, we have described the method of effectively and systematically identifying a set of potentially good designs and constructing their corresponding flexible dual contingency plans for coping with changes of resource availability levels and the unit contribution of selected produts. When a potentially good design does not satisfy the optimality condition under consideration for some change of the unit contribution of selected products, these contingency plans can convert the non-optimal solutions into optimal ones by flexibly adjusting the number of selected products made, the contribution of the products, and the units of possible slack resources. Still, many research problems remain unexplored.

For example, in addition to using the set of all potentially good designs as candidates for the final optimal designs, we can relax Assumption 1 to generate some generalized good designs by taking unions of the potentially good designs. Such generalized good designs containing a large set of products may become better candidates (see [13]). How can we construct the corresponding flexible dual contingency plans for a given generalized good design to cope with possible non-optimal situations?

In this paper, we have used a fixed adjustment level (β^0 value) to identify the flexible dual contingency plans for each potentially good design. If β value is changed, then the configuration of the flexible dual contingency plans may vary. What impact does the change of β value have on constructing the flexible dual contingency plans? Can we find a critical level of β value which guarantees to construct a set of flexible dual contingency plans with high contribution for each potentially good design?

These problems are currently under investigation and we shall report the significant results in the near future.

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